Determination of the body force of a two−dimensional isotropic elastic body

DANG DUC TRONG[a], PHAM NGOC DINH ALAIN[b],
PHAN THANH NAM[a] and TRUONG TRUNG TUYEN[c]

[a] *Mathematics Department, HoChiMinh City National University, Viet Nam*
[b] *Mathematics Department, Mapmo UMR 6628, BP 67-59, 45067 Orleans cedex, France*
[c] *Department of mathematics, Indiana University, Rawles Hall , Bloomington, IN 47405*

Abstract

Let Ω represent a two−dimensional isotropic elastic body. We consider the problem of determining the body force F whose form $\varphi(t)(f_1(x), f_2(x))$ with φ be given inexactly. The problem is nonlinear and ill-posed. Using the Fourier transform, the methods of Tikhonov's regularization and truncated integration, we construct a regularized solution from the data given inexactly and derive the explicitly error estimate.

MSC 2000: 35L20, 35R30, 42B10, 70F07, 74B05.

Key words: body force, elastic body, Fourier transform, ill−posed problem, Tikhonov's regularization, truncated integration.

1. Introduction

Let $\Omega = (0,1) \times (0,1)$ represent a two−dimensional isotropic elastic body. For each $x := (x_1, x_2) \in \Omega$, we denote by $u = (u_1(x,t),\, u_2(x,t))$ the displacement, where u_j is the displacement in the x_j- direction, for all $j \in \{1,2\}$. As known, u satisfies the Lamé system (see, e.g., [1, 2])

$$\frac{\partial^2 u}{\partial t^2} = \mu \Delta u + (\lambda + \mu)\, \nabla\, (div(u)) + F$$

where $F := (F_1, F_2)$ is the body force, $div(u) = \nabla \cdot u = \partial u_1/\partial x_1 + \partial u_2/\partial x_2$, and λ, μ are Lamé constants. We shall assume that the boundary of the elastic body is clamped and the initial conditions are given.

In this paper, we shall consider the problem of determining the body force F. The problem is a kind of inverse source problems. The inverse source problems are investigated in many aspects such as the uniqueness, the stability and the regularization. There are many papers devoted to the uniqueness and the stability problem. In [7], Isakov disscused the problem of finding a pair of functions (u, f) satisfying

$$cu_{tt} - \Delta u = f$$

where f is independent of t. He proved that using some preassumptions on f, from the *final overdetermination*

$$u(x,T) = h(x)$$

, we get the uniqueness of (u, f).

As shown in [9], the body force (in the form $\phi(t)f(x)$) will be defined uniquely from an observation of surface stress (the *lateral overdetermination*) given on a suitable boundary of $\Omega \times (0,T)$. In the paper, the authors also gave an abstract formula of reconstruction.

Another inverse source problem is one of finding the heat source $F(x,t,u)$ satisfying

$$u_t - \Delta u = F.$$

The problem was considered intensively in the last century. The problem with the final overdetermination was studied by Tikhonov in 1935 (see [8]). He proved the uniqueness of problem with prescribed lateral and final data. In the last three decades, the problem is considered by many authors (see [3, 4, 11, 12, 13, 14]). Although we have many works on the uniqueness and the stability of inverse source problems, the literature on the regularization problem is quite scarce. Very recently, in [3, 4] , the authors considered the regularization problem under both the lateral and the final overdetermination. The ideas of using the Fourier transform and truncated integration in the two papers are used in the present paper. We also consider the regularization problem under the final data and prescribed surface stress.

To get the lateral overdetermination, some mechanical arguments are in order. Let σ_1, σ_2, τ be the stresses (see [1, 2]) defined by

$$\tau = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$
$$\sigma_j = \lambda div(u) + 2\mu\frac{\partial u_j}{\partial x_j}, \qquad j \in \{1,2\}$$

We shall assume that the surface stress is given on the boundary of the body, i.e.,

$$\begin{pmatrix} \sigma_1 & \tau \\ \tau & \sigma_2 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

where $X = (X_1, X_2)$ is given on $\partial\Omega$, and $n = (n_1, n_2)$ is the outward unit normal vector of $\partial\Omega$.

As discussed, our problem is severely ill-posed. Hence, to simplify the problem, a preassumption on the form of the body force is needed. We shall use the separable form force as in [9]

$$(F_1(x,t), F_2(x,t)) = \varphi(t)(f_1(x), f_2(x))$$

where φ is given inexactly. The form is issued from an approximated model for elastic wave generated from a point dislocation source (see, e.g., [9, 10]). But, since φ is inexact, our problem is nonlinear. Morever, the problem is still ill-posed because the measured data is

not only inexact but also non-smooth.

Precisely, we consider the problem of identifying a pair of functions (u, f) satisfying the system:

$$\frac{\partial^2 u_j}{\partial t^2} = \mu \Delta u_j + (\lambda + \mu)\frac{\partial}{\partial x_j} div(u) + \varphi(t) f_j(x), \forall j \in \{1, 2\} \tag{1}$$

for $(x, t) \in \Omega \times (0, T)$, where μ, λ are real constants satisfying $\mu > 0$ and $\lambda + 2\mu > 0$. Since the boundary of the elastic body is clamped, the displacement $u = (u_1, u_2)$ satisfies the boundary condition

$$(u_1(x, t), u_2(x, t)) = (0, 0), \qquad x \in \partial\Omega \tag{2}$$

In addition, the initial and final displacement are given in Ω

$$\begin{cases} (u_1(x, 0), u_2(x, 0)) = (u_{01}(x), u_{02}(x)) \\ \left(\dfrac{\partial u_1}{\partial t}(x, 0), \dfrac{\partial u_2}{\partial t}(x, 0)\right) = (u_{01}^*(x), u_{02}^*(x)) \\ (u_1(x, T), u_2(x, T)) = (u_{T1}(x), u_{T2}(x)) \end{cases} \tag{3}$$

Finally, the surface stress is given on $\partial\Omega$

$$\begin{cases} n_1\sigma_1 + n_2\tau = X_1 \\ n_2\sigma_2 + n_1\tau = X_2 \end{cases} \tag{4}$$

We shall assume that the data of the system $(1) - (4)$

$$I = (\varphi, X, u_0, u_0^*, u_T) \in \left(L^1(0, T), (L^1(0, T, L^1(\partial\Omega)))^2, (L^1(\Omega))^2, (L^1(\Omega))^2, (L^1(\Omega))^2\right)$$

are given inexactly since they are results of experimental measurements. The system $(1)-(4)$ usually has no solution; moreover, even if the solution exists, it does not depend continously on the given data. Hence, a regularization is in order. Denoting by I_{ex} the exact data, which are probably unknown, corresponding to an exact solution (u_{ex}, f_{ex}) of the system $(1) - (4)$, from the inexact data I_ε approximating I_{ex}, we shall construct a regularized solution f_ε approximating f_{ex} .

In fact, using the Fourier transform, we shall reduce our problem to finding the solutions of the binomial equations whose binomial term is an entire function (see Lemma 1). In this case, the problem is unstable in the neighborhood of zeros of the entire function. The zeroes can be seen as singular values. Using the method of Tikhonov's regularization and truncated integration, we shall eliminate the singular values to regularize our problem. Error estimates are given.

The remainder of the paper is divided into two sections. In Section 2, we shall set some notations and state our main results. In Section 3, we give the proofs of the results.

2. Notations and main results

We recall that $\Omega = (0,1)\times(0,1)$. We always assume that the data $I = (\varphi, X, u_0, u_T, u_T^*)$ belong to

$$\left(L^1(0,T), (L^1(0,T,L^1(\partial\Omega)))^2, (L^1(\Omega))^2, (L^1(\Omega))^2, (L^1(\Omega))^2\right)$$

For all $\xi = (\xi_1, \xi_2), \zeta = (\zeta_1, \zeta_2) \in R^2$, we set $\xi\cdot\zeta = \xi_1\zeta_1 + \xi_2\zeta_2$ and $|\xi| = \sqrt{\xi\cdot\xi}$.

We first have the following lemma.

Lemma 1. *If $u \in (C^2([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)))^2$, $f \in (L^2(\Omega))^2$ satisfy (1) − (4) corresponding the data I, then for all $\alpha = (\alpha_1, \alpha_2) \in R^2\backslash\{0\}$, we have*

$$2D(I).\int\limits_\Omega f_j(x).\cos(\alpha\cdot x)dx = g_j(I), \qquad \forall j \in \{1,2\}$$

where

$$D(I) = D_1(I).D_2(I), g_j(I) = \frac{2}{|\alpha|^2}(\alpha_j D_2(I)h_0 + D_1(I)h_j)$$

with

$$D_1(I) = \int\limits_0^T \varphi(T-t)\sin(\sqrt{\lambda+2\mu}\,|\alpha|\,t)dt, D_2(I) = \int\limits_0^T \varphi(T-t)\sin(\sqrt{\mu}\,|\alpha|\,t)dt$$

$$\begin{aligned}h_0(I) = &-\sin(\sqrt{\lambda+2\mu}|\alpha|T).\int\limits_\Omega (\alpha\cdot u_0^*).\cos(\alpha\cdot x)dx\\
&+\sqrt{\lambda+2\mu}.|\alpha|.\int\limits_\Omega (\alpha\cdot u_T).\cos(\alpha\cdot x)dx\\
&-\sqrt{\lambda+2\mu}.|\alpha|.\cos(\sqrt{\lambda+2\mu}|\alpha|T).\int\limits_\Omega (\alpha\cdot u_0).\cos(\alpha\cdot x)dx\\
&-\int\limits_0^T\int\limits_{\partial\Omega} \sin(\sqrt{\lambda+2\mu}|\alpha|(T-t))(\alpha\cdot X).\cos(\alpha\cdot x)d\omega dt\end{aligned}$$

$$\begin{aligned}h_j(I) = &-\sin(\sqrt{\mu}|\alpha|T).\int\limits_\Omega (|\alpha|^2 u_{0j}^* - \alpha_j(\alpha\cdot u_0^*)).\cos(\alpha\cdot x)dx\\
&+\sqrt{\mu}.|\alpha|.\int\limits_\Omega (|\alpha|^2 u_{Tj} - \alpha_j(\alpha\cdot u_T)).\cos(\alpha\cdot x)dx\\
&-\sqrt{\mu}.|\alpha|.\cos(\sqrt{\mu}|\alpha|T).\int\limits_\Omega (|\alpha|^2 u_{0j} - \alpha_j(\alpha\cdot u_0)).\cos(\alpha\cdot x)dx\\
&-\int\limits_0^T\int\limits_{\partial\Omega} \sin(\sqrt{\mu}|\alpha|(T-t))(|\alpha|^2 X_j - \alpha_j(\alpha\cdot X)).\cos(\alpha\cdot x)d\omega dt, \forall j \in \{1,2\}.\end{aligned}$$

From Lemma 1, we consider the function

$$D(I) = \int_0^T \varphi(T-t)\sin(\sqrt{\lambda+2\mu}\,|\alpha|\,t)dt. \int_0^T \varphi(T-t)\sin(\sqrt{\mu}\,|\alpha|\,t)dt$$

The problem is unstable in the neighborhood of zeros of this function. However, from the properties of analytic function, we can show that if $\varphi \not\equiv 0$ then this function differ from 0 for almost every where in R^3. Furthermore, using the idea of Theorem 4 in [5], we get the following lemma.

Lemma 2. *Let τ, q be positive constants, $\varphi_0 \in L^1(0,T)\backslash\{0\}$ and $D(\varphi_0,\tau) : R^2 \to R$*

$$D(\varphi_0,\tau)(\alpha) = \int_0^T \varphi_0(t)\sin(\sqrt{\tau}|\alpha|t)dt$$

Then $D(\varphi_0,\tau) \neq 0$ for a.e $\alpha \in R^2$. Moreover, if we put

$$R_\varepsilon = \frac{q}{9eT}.\frac{\ln(\varepsilon^{-1})}{\ln(\ln(\varepsilon^{-1}))}, \qquad \forall \varepsilon > 0$$

then the Lebesgue measure of the set

$$B(\varphi_0,\tau,\varepsilon) = \{\alpha \in B(0,R_\varepsilon), |D(\varphi_0,\tau)(\alpha)| \leq \varepsilon^q\}$$

is less than R_ε^{-1} for $\varepsilon > 0$ small enough, where $B(0,R_\varepsilon)$ is the open ball in R^2 .

Lemma 1 and Lemma 2 imply immediately the uniqueness result.

Theorem 1. *Let $u, u^* \in (C^2([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)))^2$, $f, f^* \in (L^2(\Omega))^2$. If (u,f), (u^*,f^*) satisfy $(1)-(4)$ corresponding the same data I, and $\varphi \not\equiv 0$, then*

$$(u,f) = (u^*,f^*)$$

Let (u_{ex}, f_{ex}) be the exact solution of the system $(1)-(4)$ corresponding the exact data $I_{ex} = (\varphi_{ex}, X_{ex}, u_0^{ex}, u_0^{*ex}, u_T^{ex})$. Notice that, if we assume

$$u_{ex} \in (C^2([0,T];L^2(\Omega)) \cap L^2([0,T];H^2(\Omega)))^2, f_{ex} \in (L^2(\Omega))^2, \varphi_{ex} \in L^1(0,T)\backslash\{0\} \qquad (5)$$

then for all $j \in \{1,2\}$,

$$F(\widetilde{f_{jex}})(\alpha) = 2\int_\Omega f_{jex}(x)\cos(\alpha\cdot x)dx = \frac{g_j(I_{ex})}{D(I_{ex})}$$

for a.e $\alpha \in R^2$, where g_j, D are defined by Lemma 1, $\widetilde{f}_{jex} : R^2 \to R$ is defined by $\widetilde{f}_{jex}(x) = \chi(\Omega) f_{jex}(x) + \chi(-\Omega) f_{jex}(-x)$, and F is the Fourier transform in R^2.

From approximate data $I_\varepsilon = (\varphi, X, u_0, u_0^*, u_T)$ satisfying

$$\begin{aligned}
&\|\varphi - \varphi_{ex}\|_{L^1(0,T)} \le \varepsilon, \left\|X_j - X_j^{ex}\right\|_{L^1(0,T,L^1(\partial\Omega))} \le \varepsilon, \left\|u_{0j} - u_{0j}^{ex}\right\|_{L^1(\Omega)} \le \varepsilon \\
&\left\|u_{0j}^* - u_{0i}^{*ex}\right\|_{L^1(\Omega)} \le \varepsilon, \left\|u_{Tj} - u_{Tj}^{ex}\right\|_{L^1(\Omega)} \le \varepsilon, \qquad \forall j \in \{1,2\}
\end{aligned} \tag{6}$$

, we construct a regularized solution $f_\varepsilon = (f_{1\varepsilon}, f_{2\varepsilon})$ whose Fourier transform is

$$F(f_{j\varepsilon})(\alpha) = \chi(B(0,R_\varepsilon)).\frac{g_j(I_\varepsilon).D(I_\varepsilon)}{\delta_\varepsilon + (D(I_\varepsilon))^2}, \forall \alpha \in R^2 \backslash \{0\}$$

where

$$q = \frac{1}{7}, \delta_\varepsilon = \varepsilon^{\frac{1+6q}{2}}, R_\varepsilon = \frac{q}{9eT}.\frac{ln(\varepsilon^{-1})}{ln(ln(\varepsilon^{-1}))} \tag{7}$$

We have two regularization results.

Theorem 2. *Let (u_{ex}, f_{ex}) be the exact solution of the system $(1)-(4)$ corresponding the exact data I_{ex}, and (5) hold. Then from the given data I_ε satisfying (6), we can construct a regularized solution $f_\varepsilon \in (C(\overline{\Omega}))^2$ such that*

$$\lim_{\varepsilon \to 0} \|f_{j\varepsilon} - f_{jex}\|_{L^2(\Omega)} = 0, \qquad \forall j \in \{1,2\}$$

If we assume, in addition, that $f_{ex} \in (H^1(\Omega))^2$, then

$$\|f_{j\varepsilon} - f_{jex}\|^2_{L^2(\Omega)} \le 63eT\left(66\,\|f_{jex}\|^2_{H^1(\Omega)} + (2\pi)^{-2}\right).\frac{\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}, \qquad \forall j \in \{1,2\}$$

for $\varepsilon > 0$ small enough.

Theorem 3. *Let (u_{ex}, f_{ex}) be the exact solution of the system $(1)-(4)$ corresponding the exact data I_{ex}, and (5) hold. We assume, in addition, that*

$$\int_{R^2} \left| \int_{\Omega} f_{jex}(x).\cos(\alpha \cdot x) dx \right| d\alpha < \infty, \qquad \forall j \in \{1,2\}$$

Then from the given data I_ε satisfying (6), we can construct a regularized solution $f_\varepsilon \in (C(\overline{\Omega}))^2$, which coincides the one in Theorem 2, such that

$$\lim_{\varepsilon \to 0} \|f_{j\varepsilon} - f_{jex}\|_{L^\infty(\Omega)} = 0, \qquad \forall j \in \{1,2\}$$

3. Proofs of the results

Proof of Lemma 1

Proof. Let $\alpha = (\alpha_1, \alpha_2) \in R^2$ and $G = \cos(\alpha \cdot x)$. Notice that the j−th equation of the system (1) can rewrite

$$\frac{\partial^2 u_j}{\partial t^2} = \frac{\partial \sigma_j}{\partial x_j} + \frac{\partial \tau}{\partial x_k} + \varphi(t) f_j(x), \qquad \{j,k\} = \{1,2\}$$

Getting the inner product (in $L^2(\Omega)$) of the equation and G and using the condition (2), for $\{j,k\} = \{1,2\}$, we get

$$\begin{aligned} \frac{d}{dt^2} \int\limits_{\Omega} u_j G &= \int\limits_{\partial\Omega} (n_j \sigma_j + n_k \tau) G d\omega - \int\limits_{\Omega} \sigma_j \frac{\partial G}{\partial x_j} dx - \int\limits_{\Omega} \tau \frac{\partial G}{\partial x_k} dx + \varphi(t) \int\limits_{\Omega} f_j G dx \\ &= \int\limits_{\partial\Omega} X_j G d\omega - \mu |\alpha|^2 \int\limits_{\Omega} u_j G dx - (\lambda + \mu)\alpha_j \int\limits_{\Omega} (\alpha \cdot u) G dx + \varphi(t) \int\limits_{\Omega} f_j G dx \end{aligned} \tag{8}$$

Multiplying (8) by α_j, then getting the sum for $j = 1, 2$, we obtain

$$\frac{d}{dt^2} \int\limits_{\Omega} (\alpha \cdot u) G dx = \int\limits_{\partial\Omega} (\alpha \cdot X) G d\omega - (\lambda + 2\mu)|\alpha|^2 \int\limits_{\Omega} (\alpha \cdot u) G dx + \varphi(t) \int\limits_{\Omega} (\alpha \cdot f) G dx \tag{9}$$

Multiplying (8) by $|\alpha|^2$ and multiplying (9) by $-\alpha_j$, then getting the sum of them, we have

$$\begin{aligned} \frac{d}{dt^2} \int\limits_{\Omega} \left(|\alpha|^2 u_j - \alpha_j . (\alpha \cdot u) \right) G dx &= \int\limits_{\partial\Omega} \left(|\alpha|^2 X_j - \alpha_j . (\alpha \cdot X) \right) G dx \\ -\mu |\alpha|^2 \int\limits_{\Omega} \left(|\alpha|^2 u_j - \alpha_j . (\alpha \cdot u) \right) G dx &+ \varphi(t) \int\limits_{\Omega} \left(|\alpha|^2 f_j - \alpha_j . (\alpha \cdot f) \right) G dx \end{aligned} \tag{10}$$

We consider (9) and (10) as the differential equations whose form

$$y'' + \eta^2 y = h(t) \tag{11}$$

where η is a real constant and $y(0)$, $y'(0)$, $y(T)$ are given. Getting the inner product (in $L^2(0,T)$) of (11) and $\sin(\eta(T-t))$, we have

$$-y'(0) sin(\eta T) + \eta y(T) - \eta y(0) cos(\eta T) = \int\limits_0^T h(T-t) \sin(\eta t) dt \tag{12}$$

Applying (12) to (9) with $\eta = \sqrt{(\lambda + 2\mu)}|\alpha|$ and $y = \int\limits_{\Omega} (\alpha \cdot u) . G dx$, we get

$$D_1(I) . \int\limits_{\Omega} (\alpha \cdot f) . G dx = h_0(I) \tag{13}$$

where $D_1(I), h_0(I)$ are defined by Lemma 1.
Similarly, applying (12) to (10) with $\eta = \sqrt{\mu}|\alpha|$ and $y = \int\limits_{\Omega} (|\alpha|^2 u_j - \alpha_j.(\alpha \cdot u)).Gdx$, we get

$$D_2(I). \int\limits_{\Omega} (|\alpha|^2 f_j - \alpha_j(\alpha \cdot f)).Gdx = h_j(I), \qquad \forall j \in \{1,2\} \tag{14}$$

where $D_2(I), h_j(I)$ are defined by Lemma 1.

Multiplying (13) by $\alpha_j D_2(I)$ and multiplying (14) by $D_1(I)$, then getting the sum of them, we obtain the result of Lemma 1. □

Proof of Lemma 2

Proof. Put $\widetilde{\varphi}_0 : R \to R$

$$\widetilde{\varphi}_0(t) = \frac{1}{2}\begin{cases} \varphi_0(t) & t \in (0,T) \\ -\varphi_0(-t) & t \in (-T,0) \\ 0 & t \notin (-T,T) \end{cases}$$

and $\phi : C \to C$

$$\phi(z) = \int\limits_{-\infty}^{\infty} e^{-itz}\widetilde{\varphi}_0(t)dt = \int\limits_{-T}^{T} e^{-itz}\widetilde{\varphi}_0(t)dt$$

Then ϕ is an entire function and $D(\varphi_0,\tau)(\alpha) = i\phi(\sqrt{\tau}|\alpha|)$. Because $\widetilde{\varphi}_0 \not\equiv 0$, its Fourier transform (in R) does not coincide 0. Therefore, there exists $z_0 \in R$ such that $|\phi(z_0)| = C_1 > 0$. Thus $\phi \not\equiv 0$. Since ϕ is an entire function, its zeros set is either finite or countable. Consequently, $D(\varphi_0,\tau)(\alpha) \neq 0$ for a.e $\alpha \in R^2$.

To estimate the measure of $B(\varphi_0,\tau,\varepsilon)$, we shall use the following result (see Theorem 4 of \$11.3 in [6]).

Lemma 3. *Let $f(z)$ be a function analytic in the disk $\{z : |z| \leq 2eR\}$, $|f(0)| = 1$, and let η be an arbitrary small positive number. Then the estimate*

$$\ln|f(z)| > -\ln(\frac{15e^3}{\eta}).\ln(M_f(2eR))$$

is valid everywhere in the disk $\{z : |z| \leq R\}$ except a set of disks (C_j) with sum of radii $\sum r_j \leq \eta R$. Where $M_f(r) = \max\limits_{|z|=r}|f(z)|$.

Returning Lemma 2, we put $\phi_1 : C \to C$

$$\phi_1(z) = \frac{\phi(z+z_0)}{C_1}$$

Then ϕ_1 is an entire function, $\phi_1(0) = 1$, and for all $z \in C, |z| \le 2eR$,

$$C_1 |\phi_1(z)| = \left| \int_{-T}^{T} e^{-it(z+z_0)} \widetilde{\varphi}_0(t) \right| \le e^{2eRT} . \int_{-T}^{T} |\widetilde{\varphi}_0(t)| \, dt = e^{2eRT} \|\varphi_0\|_{L^1(0,T)}$$

For $\varepsilon > 0$ small enough, applying Lemma 3 with $R = \frac{4}{3} R_\varepsilon$ and $\eta = \frac{\sqrt{\tau}}{8\pi R_\varepsilon^3}$, we get

$$\begin{aligned} \ln |\phi_1(z)| &> - \left[3 \ln R_\varepsilon + \ln(\frac{8\pi}{\sqrt{\tau}}) + \ln(15e^3) \right] . \left[\frac{8}{3} . eTR_\varepsilon + \ln\left(\frac{\|\varphi_0\|_{L^1(0,T)}}{C_1} \right) \right] \\ &> -\frac{17}{2} T . R_\varepsilon \ln R_\varepsilon > -q \ln(\varepsilon^{-1}) - \ln(C_1) = \ln(\frac{\varepsilon^q}{C_1}) \end{aligned}$$

for all $|z| \le \frac{4}{3} R_\varepsilon$ except a set of disks $\{B(z_j, r_j)\}_{j \in J}$ with sum of radii $\sum r_i \le \eta R = \frac{\sqrt{\tau}}{6\pi R_\varepsilon^2}$.

Consequently, for $\varepsilon > 0$ small enough, we have $|z_0| < \frac{1}{3} R_\varepsilon$ and $|\phi(z)| = C_1 . |\phi_1(z - z_0)| \ge \varepsilon^q$ for all $|z| \le R_\varepsilon$ except the set $\underset{j \in J}{\cup} B(z_j + z_0, r_j)$. Hence, $B(\varphi_0, \tau, \varepsilon)$ is contained in the set $\underset{j \in J}{\cup} B_j$, where

$$B_j = \{\alpha \in B(0, R_\varepsilon), \left| \sqrt{\tau} |\alpha| - y_j \right| \le r_j\}$$

with $y_j = Re(z_j + z_0)$.

If $y_j > \sqrt{\tau} R_\varepsilon + r_j$ then $B_j = \emptyset$. If $y_j \le r_j$ then $B_j \subset B(0, \frac{2r_j}{\sqrt{\tau}})$, so $m(B_j) \le \frac{4\pi r_j^2}{\tau}$. If $r_j < y_j \le \sqrt{\tau} R_\varepsilon + r_j$ then

$$B_j \subset B(0, \frac{y_j + r_j}{\sqrt{\tau}}) \backslash B(0, \frac{y_j - r_j}{\sqrt{\tau}})$$

hence

$$m(B_j) \le \frac{\pi (y_j + r_j)^2}{\tau} - \frac{\pi (y_j - r_j)^2}{\tau} = \frac{4\pi y_j r_j}{\tau} \le \frac{4\pi (\sqrt{\tau} R_\varepsilon + r_j) r_j}{\tau}$$

Thus we get

$$\begin{aligned} m(B(\varphi, \tau, \varepsilon)) &\le \sum \frac{4\pi (\sqrt{\tau} R_\varepsilon + r_j) r_j}{\tau} + \sum \frac{4\pi r_j^2}{\tau} \\ &\le \frac{4\pi R_\varepsilon}{\sqrt{\tau}} \sum r_j + \frac{8\pi}{\tau} (\sum r_j)^2 \le \frac{4\pi R_\varepsilon}{\sqrt{\tau}} . \frac{\sqrt{\tau}}{6\pi R_\varepsilon^2} + \frac{8\pi}{\tau} . (\frac{\sqrt{\tau}}{6\pi R_\varepsilon^2})^2 < \frac{1}{R_\varepsilon} \end{aligned}$$

for $\varepsilon > 0$ small enough. The proof of Lemma 2 is completed. $\square$

Proof of theorem 1

Proof. Put $w = u - u^*$ and $v = f - f^*$ then (w, v) satisfies $(1) - (4)$ corresponding the data

$$I = (\varphi, (0,0), (0,0), (0,0), (0,0))$$

Let $\widetilde{v}_j : R^2 \to R$ be defined by $\widetilde{v}_j(x) = \chi(\Omega) v_j(x) + \chi(-\Omega) v_j(-x)$. Lemma 1 implies that, for all $j \in \{1, 2\}$, for all $\alpha \in R^2 \backslash \{0\}$, we get

$$D(I).F(\widetilde{v}_j)(\alpha) = 2D(I). \int\limits_{\Omega} v_j(x) \cos(\alpha \cdot x) dx = g_j(I) = 0$$

Applying Lemma 2 with $\varphi_0(t) = \varphi(T - t)$, we get $D(I) \neq 0$ for a.e $\alpha \in R^2$. Therefore, $F(\widetilde{v}_j) \equiv 0$, and it implies that $\widetilde{v}_j \equiv 0$. Thus $v \equiv (0, 0)$. Hence, w satisfies that

$$\frac{\partial^2 w}{\partial t^2} = \mu \Delta w + (\lambda + \mu) \nabla (div(w)) \tag{15}$$

Getting the inner product (in $(L^2(\Omega))^2$) of (15) and $\partial w / \partial t$, we have

$$\frac{1}{2} . \frac{d}{dt} \sum_{j=1}^{2} \left\| \frac{\partial w_j}{\partial t} \right\|_{L^2(\Omega)}^2 = -\frac{\mu}{2} . \frac{d}{dt} \sum_{j=1}^{2} \|\nabla w_j\|_{L^2(\Omega)}^2 - \frac{\lambda + \mu}{2} . \frac{d}{dt} \|div(w)\|_{L^2(\Omega)}^2$$

Integrating this equality in $(0, t)$, we get

$$\sum_{j=1}^{2} \left\| \frac{\partial w_j}{\partial t} \right\|_{L^2(\Omega)}^2 + \mu \sum_{j=1}^{2} \|\nabla w_j\|_{L^2(\Omega)}^2 + (\lambda + \mu) \|div(w)\|_{L^2(\Omega)}^2 = 0 \tag{16}$$

for all $t \in (0, T)$. Using the condition (2), we have

$$\begin{aligned} \|div(w)\|_{L^2(\Omega)}^2 &= \sum_{j=1}^{2} \left\| \frac{\partial w_j}{\partial x_j} \right\|_{L^2(\Omega)}^2 + 2 \int\limits_{\Omega} \frac{\partial w_1}{\partial x_1} . \frac{\partial w_2}{\partial x_2} = \sum_{j=1}^{2} \left\| \frac{\partial w_j}{\partial x_j} \right\|_{L^2(\Omega)}^2 + 2 \int\limits_{\Omega} \frac{\partial w_1}{\partial x_2} . \frac{\partial w_2}{\partial x_1} \\ &\leq \sum_{j=1}^{2} \left\| \frac{\partial w_j}{\partial x_j} \right\|_{L^2(\Omega)}^2 + \left(\left\| \frac{\partial w_1}{\partial x_2} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial w_2}{\partial x_1} \right\|_{L^2(\Omega)}^2 \right) = \sum_{j=1}^{2} \|\nabla w_j\|_{L^2(\Omega)}^2 \end{aligned}$$

Since $\mu > 0$ and $\lambda + 2\mu > 0$, the above inequality implies that

$$\mu \sum_{j=1}^{2} \|\nabla w_j\|_{L^2(\Omega)}^2 + (\lambda + \mu) \|div(w)\|_{L^2(\Omega)}^2 \geq 0$$

From (16), we obtain $\partial w / \partial t = (0, 0)$. Since $w(x, 0) = (0, 0)$, the proof is completed. $\square$

To prove two main regularization results, we state and prove some preliminary lemmas.

Lemma 4. *Let (u_{ex}, f_{ex}) be the exact solution of* $(1)-(4)$ *corresponding the exact data I_{ex} satisfying* (5)*, and the given data I_ε satisfying* (6)*. Using notations of* (7)*, we put*

$$G_j(I_\varepsilon) = \chi(B(0,R_\varepsilon)).\frac{g_j(I_\varepsilon)D(I_\varepsilon)}{\delta_\varepsilon + (D(I_\varepsilon))^2}$$

Then for all $j \in \{1,2\}$, we have $G_j(I_\varepsilon) \in L^1(R^2) \cap L^2(R^2)$; moreover, there exists a constant C_0 depend only on I_{ex} such that for all $\varepsilon \in (0, e^{-e})$,

$$\left|G_j(I_\varepsilon) - F(\widetilde{f}_{jex})\right| \le \chi(B(0,R_\varepsilon))C_0 R_\varepsilon \varepsilon^{\frac{1-6q}{2}}$$
$$+2\chi(B_\varepsilon)\left\|f_{jex}\right\|_{L^2(\Omega)} + \chi(R^2\backslash B(0,R_\varepsilon))\left|F(\widetilde{f}_{jex})\right|$$

where $B_\varepsilon = \left\{\alpha \in B(0,R_\varepsilon), |D(I_{ex})(\alpha)| \le \varepsilon^{2q}\right\}$.

Proof. First, we show that there exists a constant $C_2 > 0$ depend only on I_{ex} such that for all $\varepsilon \in (0, e^{-e})$, $r > r_0 = q/(9T)$, $j \in \{1,2\}$,

$$\|D(I_{ex})\|_{L^\infty(R^2)} \le C_2, \|D(I_\varepsilon) - D(I_{ex})\|_{L^\infty(R^2)} \le C_2\varepsilon$$
$$\|g_j(I_{ex})\|_{L^\infty(B(0,r))} \le C_2 r, \|g_j(I_\varepsilon) - g_j(I_{ex})\|_{L^\infty(B(0,r))} \le C_2 r\varepsilon$$

Recall that $D_1(I), D_2(I), h_0(I), h_j(I)$ are defined by Lemma 1. For all $\alpha \in R^3$ we have

$$|D_k(I_{ex})| \le \|\varphi_{ex}\|_{L^1(0,T)}, |D_k(I_\varepsilon) - D_k(I_{ex})| \le \|\varphi_\varepsilon - \varphi_{ex}\|_{L^1(0,T)} \le \varepsilon$$

for all $k \in \{1,2\}$. Hence, $|D(I_{ex})| \le \|\varphi_{ex}\|^2_{L^1(0,T)}$ and

$$|D(I_\varepsilon) - D(I_{ex})| = |D_1(I_\varepsilon) - D_1(I_{ex})|.|D_2(I_\varepsilon)| + |D_1(I_{ex})|.|D_2(I_\varepsilon) - D_2(I_{ex})|$$
$$\le \varepsilon.(\|\varphi_{ex}\|_{L^1(0,T)} + \varepsilon) + \|\varphi_{ex}\|_{L^1(0,T)}.\varepsilon \le (2\|\varphi_{ex}\|_{L^1(0,T)} + e^{-e}).\varepsilon$$

A straightforward calculation show that, for all $\alpha \in B(0,r)\backslash\{0\}$, we have

$$|\alpha_j h_0(I_{ex})| \le C_3 r|\alpha|^2, |\alpha_j(h_0(I_\varepsilon) - h_0(I_{ex}))| \le C_3 r|\alpha|^2\varepsilon,$$
$$|h_j(I_{ex})| \le C_3 r|\alpha|^2, |h_j(I_\varepsilon) - h_j(I_{ex})| \le C_3 r|\alpha|^2\varepsilon$$

for all $j \in \{1,2\}$, where C_3 is a positive constant depending only on I_{ex}. Therefore,

$$|g_j(I_{ex})| \le \frac{|\alpha_j h_0(I_{ex})|}{|\alpha|^2}.|D_2(I_{ex})| + \frac{|h_j(I_{ex})|}{|\alpha|^2}.|D_1(I_{ex})| \le 2C_3\|\varphi_{ex}\|_{L^1(0,T)} r$$

and

$$|g_j(I_\varepsilon) - g_j(I_{ex})| \le \frac{|\alpha_j(h_0(I_\varepsilon) - h_0(I_{ex}))|}{|\alpha|^2}.|D_2(I_\varepsilon)| + \frac{|\alpha_j h_0(I_{ex})|}{|\alpha|^2}.|D_2(I_\varepsilon) - D_2(I_{ex})|$$
$$+\frac{|h_j(I_\varepsilon) - h_0(I_{ex})|}{|\alpha|^2}.|D_1(I_\varepsilon)| + \frac{|h_j(I_{ex})|}{|\alpha|^2}.|D_1(I_\varepsilon) - D_1(I_{ex})|$$
$$\le C_3 r\varepsilon.\left(\|\varphi_{ex}\|^2_{L^1(0,T)} + \varepsilon\right) + C_3 r.\varepsilon + C_3 r\varepsilon.\left(\|\varphi_{ex}\|^2_{L^1(0,T)} + \varepsilon\right) + C_3 r.\varepsilon$$
$$\le 2C_3\left(\|\varphi_{ex}\|^2_{L^1(0,T)} + e^{-e} + 1\right)r\varepsilon$$

Returning Lemma 4, for all $j \in \{1,2\}$, we get $G_j(I_\varepsilon) \in L^1(R^2) \cap L^2(R^2)$ because the support of $G_j(I_\varepsilon)$ is contained in $\overline{B(0,R_\varepsilon)}$ and $G_j(I_\varepsilon) \in L^\infty(R^2)$. Moreover,

$$\left|G_j(I_\varepsilon) - F(\widetilde{f}_{jex})\right| \leq \chi(B(0,R_\varepsilon)) \left|\frac{g_j\left(I_\varepsilon\right) D(I_\varepsilon)}{\delta_\varepsilon + \left(D(I_\varepsilon)\right)^2} - \frac{g_j\left(I_{ex}\right) D(I_{ex})}{\delta_\varepsilon + \left(D(I_{ex})\right)^2}\right|$$
$$+\chi(B(0,R_\varepsilon)) \left|\frac{g_j\left(I_{ex}\right) D(I_{ex})}{\delta_\varepsilon + \left(D(I_{ex})\right)^2} - \frac{g_j\left(I_{ex}\right)}{D(I_{ex})}\right| + \chi(R^2 \backslash B(0,R_\varepsilon)). \left|F(\widetilde{f}_{jex})\right|$$

We shall estimate each of the terms of the right-hand side. We have

$$\left|\frac{g_j\left(I_\varepsilon\right) D(I_\varepsilon)}{\delta_\varepsilon + \left(D(I_\varepsilon)\right)^2} - \frac{g_j\left(I_{ex}\right) D(I_{ex})}{\delta_\varepsilon + \left(D(I_{ex})\right)^2}\right| \leq \frac{\delta_\varepsilon \left|g_j\left(I_\varepsilon\right) D(I_\varepsilon) - g_j\left(I_{ex}\right) D(I_{ex})\right|}{\left(\delta_\varepsilon + \left(D(I_\varepsilon)\right)^2\right)\left(\delta_\varepsilon + \left(D(I_{ex})\right)^2\right)}$$
$$+\frac{\left|D(I_\varepsilon)\right| . \left|D(I_{ex})\right| . \left|g_j\left(I_\varepsilon\right) D(I_{ex}) - g_j\left(I_{ex}\right) D(I_\varepsilon)\right|}{\left(\delta_\varepsilon + \left(D(I_\varepsilon)\right)^2\right)\left(\delta_\varepsilon + \left(D(I_{ex})\right)^2\right)}$$
$$\leq \frac{\left|g_j\left(I_\varepsilon\right) D(I_\varepsilon) - g_j\left(I_{ex}\right) D(I_{ex})\right|}{\delta_\varepsilon} + \frac{\left|g_j\left(I_\varepsilon\right) D(I_{ex}) - g_j\left(I_{ex}\right) D(I_\varepsilon)\right|}{\delta_\varepsilon}$$

If $\varepsilon \in (0, e^{-e})$ then $R_\varepsilon > r_0$, so for all $\alpha \in B(0,R_\varepsilon)$ we get

$$\begin{aligned}
&|g_j(I_\varepsilon)D(I_\varepsilon) - g_j(I_{ex})D(I_{ex})| \\
&\leq |g_j(I_\varepsilon) - g_j(I_{ex})| . |D(I_\varepsilon)| + |g_j(I_{ex})| . |D(I_\varepsilon) - D(I_{ex})| \\
&\leq C_2 R_\varepsilon \varepsilon . (C_2 + \varepsilon) + C_2 R_\varepsilon \varepsilon \leq (C_2+1)^2 R_\varepsilon \varepsilon
\end{aligned}$$

and similarly,

$$|g_j(I_\varepsilon)D(I_{ex}) - g_j(I_{ex})D(I_\varepsilon)| \leq (C_2+1)^2 R_\varepsilon \varepsilon$$

Consequently, for all $\varepsilon \in (0, e^{-e})$, we can estimate the first term

$$\chi(B(0,R_\varepsilon)) \left|\frac{g_j\left(I_\varepsilon\right) D(I_\varepsilon)}{\delta_\varepsilon + \left(D(I_\varepsilon)\right)^2} - \frac{g_j\left(I_{ex}\right) D(I_{ex})}{\delta_\varepsilon + \left(D(I_{ex})\right)^2}\right| \leq \chi(B(0,R_\varepsilon)). \frac{2(C_2+1)^2 R_\varepsilon \varepsilon}{\delta_\varepsilon}$$

Considering the second term, we have

$$\left|\frac{g_j\left(I_{ex}\right) D(I_{ex})}{\delta_\varepsilon + \left(D(I_{ex})\right)^2} - \frac{g_j\left(I_{ex}\right)}{D(I_{ex})}\right| = \frac{\delta_\varepsilon \left|g_j\left(I_{ex}\right)\right|}{\left(\delta_\varepsilon + \left(D(I_{ex})\right)^2\right) . \left|D(I_{ex})\right|}$$

We always have

$$\frac{\delta_\varepsilon \left|g_j\left(I_{ex}\right)\right|}{\left(\delta_\varepsilon + \left(D(I_{ex})\right)^2\right) . \left|D(I_{ex})\right|} \leq \left|\frac{g_j\left(I_{ex}\right)}{D(I_{ex})}\right| = 2\left|\int\limits_\Omega f_{jex}(x)\cos(\alpha \cdot x)dx\right| \leq 2\left\|f_{jex}\right\|_{L^2(\Omega)}$$

Furthermore, if $\alpha \in B(0,R_\varepsilon) \backslash B_\varepsilon$ then

$$\frac{\delta_\varepsilon \left|g_j\left(I_{ex}\right)\right|}{\left(\delta_\varepsilon + \left(D(I_{ex})\right)^2\right) . \left|D(I_{ex})\right|} \leq \frac{\delta_\varepsilon \left|g_j\left(I_{ex}\right)\right|}{\left|D(I_{ex})\right|^3} \leq \frac{\delta_\varepsilon C_2 R_\varepsilon}{\varepsilon^{6q}}$$

Therefore, for all $\varepsilon \in (0, e^{-e})$, we can estimate the second term

$$\chi(B(0,R_\varepsilon))\left|\frac{g_j\left(I_{ex}\right)D(I_{ex})}{\delta_\varepsilon+\left(D(I_{ex})\right)^2}-\frac{g_j\left(I_{ex}\right)}{D(I_{ex})}\right|\le 2\chi(B_\varepsilon)\left\|f_{jex}\right\|_{L^2(\Omega)}+\chi(B(0,R_\varepsilon))\frac{\delta_\varepsilon C_2 R_\varepsilon}{\varepsilon^{6q}}$$

Thus, for all $\varepsilon \in (0, e^{-e})$, we have

$$\begin{aligned}\left|G_j(I_\varepsilon)-F(\widetilde{f}_{jex})\right| &\le \chi(B(0,R_\varepsilon))\left(\frac{2(C_2+1)^2R_\varepsilon\varepsilon}{\delta_\varepsilon}+\frac{\delta_\varepsilon C_2 R_\varepsilon}{\varepsilon^{6q}}\right)\\ &+2\chi(B_\varepsilon)\left\|f_{jex}\right\|_{L^2(\Omega)}+\chi(R^2\backslash B(0,R_\varepsilon))\left|F(\widetilde{f}_{jex})\right|\end{aligned}$$

Choosing $\delta_\varepsilon = \varepsilon^{\frac{6q+1}{2}}$ and $C_0 = 2(C_2+1)^2 + C_2$, we complete the proof. $\square$

It is obvious that, for all $j \in \{1, 2\}$, by Lebesgue's dominated convergence theorem, $\chi(R^2\backslash B(0,R_\varepsilon))\left|F(\widetilde{f}_{jex})\right|$ converges to 0 in $L^2(R^2)$ when $\varepsilon \to 0$. However, to get an explicitly estimate for it, some a-priori information about f_{ex} must be assume.

Lemma 5. *Let $a \in R$, Q be an measurable subset of R^n $(n \geq 1)$, and $w \in L^1(Q) \cap L^2(Q)$. Then*

$$\int_{R^n}\left|\int_Q w(x)\sin(a+\sum_{k=1}^n \alpha_k x_k)dx\right|^2 d\alpha = 2^{n-1}\pi^n\left\|w\right\|^2_{L^2(Q)}$$

Proof. We first prove in the case $a = 0$. Put $\widetilde{w} : R^n \to R$

$$\widetilde{w}(x) = \chi(Q)w(x) - \chi(-Q)w(-x)$$

Then $\widetilde{w} \in L^1(R^n) \cap L^2(R^n)$ and

$$F_n(\widetilde{w})(\alpha) = 2i\int_Q w(x)\sin(\sum_{k=1}^n \alpha_k x_k)dx$$

where F_n is the Fourier transform in R^n. Using Paserval equality, we get

$$\int_{R^n}\left|\int_Q w(x)\sin(\sum_{k=1}^n \alpha_k x_k)dx\right|^2 d\alpha = \frac{1}{4}\left\|F_n(\widetilde{w})\right\|^2_{L^2(R^n)} = \frac{(2\pi)^n}{4}\left\|\widetilde{w}\right\|^2_{L^2(R^n)} = 2^{n-1}\pi^n\left\|w\right\|^2_{L^2(Q)}$$

Similarly, we also have

$$\int_{R^n}\left|\int_Q w(x)\cos(\sum_{k=1}^n \alpha_k x_k)dx\right|^2 d\alpha = 2^{n-1}\pi^n\left\|w\right\|^2_{L^2(Q)}$$

Now, we notice that

$$\left|\int_Q w(x)\sin(a+\sum_{k=1}^n \alpha_k x_k)dx\right|^2 = (\cos(a))^2\left|\int_Q w(x)\sin(\sum_{k=1}^n \alpha_k x_k)dx\right|^2$$
$$+(\sin(a))^2\left|\int_Q w(x)\cos(\sum_{k=1}^n \alpha_k x_k)dx\right|^2 + v(\alpha)$$

where

$$v(\alpha) = \sin(2a).\int_\Omega w(x)\sin(\sum_{k=1}^n \alpha_k x_k)dx.\int_\Omega w(x)\cos(\sum_{k=1}^n \alpha_k x_k)dx$$

Since $v(-\alpha) = -v(\alpha)$ for all $\alpha \in R^n$, we get $\int_{R^n} v(\alpha)d\alpha = 0$. Thus

$$\int_{R^n}\left|\int_Q w(x)\sin(a+\sum_{k=1}^n \alpha_k x_k)dx\right|^2 d\alpha$$
$$= (\cos(a))^2.2^{n-1}\pi^n\|w\|^2_{L^2(Q)} + (\sin(a))^2.2^{n-1}\pi^n\|w\|^2_{L^2(Q)} = 2^{n-1}\pi^n\|w\|^2_{L^2(Q)}$$

The proof is completed. □

Using Lemma 5, we have the following result.

Lemma 6. *Let $w \in H^1(\Omega)$ and $r > \pi/(2\sqrt{2})$. Then*

$$\int_{R^2\setminus B(0,r)}\left|\int_\Omega w(x)\cos(\alpha\cdot x)dx\right|^2 d\alpha \le \frac{72\sqrt{2}\pi}{r}\|w\|^2_{H^1(\Omega)}$$

Proof. Since

$$\int_{R^2\setminus B(0,r)}\left|\int_\Omega w(x)\cos(\alpha\cdot x)dx\right|^2 d\alpha \le \sum_{j=1}^2\int_{|\alpha_j|\ge r/\sqrt{2}}\left|\int_\Omega w(x)\cos(\alpha\cdot x)dx\right|^2 d\alpha$$

, the proof will be completed if we show that, for all $j \in \{1,2\}$,

$$\int_{|\alpha_j|\ge r/\sqrt{2}}\left|\int_\Omega w(x)\cos(\alpha\cdot x)dx\right|^2 d\alpha \le \frac{24\sqrt{2}\pi}{r}\left(\|w\|^2_{L^2(\Omega)} + 2\left\|\frac{\partial w}{\partial x_j}\right\|^2_{L^2(\Omega)}\right)$$

We will prove for the case $j = 1$, and the other cases are similar. We have

$$\int\limits_{\Omega} w(x)\cos(\alpha\cdot x)dx = \int\limits_{0}^{1}\left[w(x)\frac{\sin(\alpha\cdot x)}{\alpha_1}\right]_{x_1=0}^{x_1=1} dx_2 - \int\limits_{\Omega}\frac{\partial w}{\partial x_1}.\frac{\sin(\alpha\cdot x)}{\alpha_1}dx$$

so

$$\left|\int\limits_{\Omega} w(x)\cos(\alpha\cdot x)dx\right|^2 \leq \frac{3}{\alpha_1^2}\left|\int\limits_{0}^{1} w(1,x_2)\sin(\alpha_1+\alpha_2 x_2)dx_2\right|^2$$

$$+\frac{3}{\alpha_1^2}\left|\int\limits_{0}^{1} w(0,x_2)\sin(\alpha_2 x_2)dx_2\right|^2 + \frac{3}{\alpha_1^2}\left|\int\limits_{\Omega}\frac{\partial w}{\partial x_1}.\sin(\alpha\cdot x)dx\right|^2$$

Therefore,

$$\int\limits_{|\alpha_1|\geq r/\sqrt{2}}\left|\int\limits_{\Omega} w(x)\cos(\alpha\cdot x)dx\right|^2 d\alpha \leq \frac{6}{r^2}\int\limits_{R^2}\left|\int\limits_{\Omega}\frac{\partial w}{\partial x_1}(x).\sin(\alpha\cdot x)dx\right|^2 d\alpha$$

$$+\int\limits_{|\alpha_1|\geq r/\sqrt{2}}\frac{3}{\alpha_1^2}d\alpha_1.\int\limits_{-\infty}^{\infty}\left|\int\limits_{0}^{1} w(1,x_2)\sin(\alpha_1+\alpha_2 x_2)dx_2\right|^2 d\alpha_2$$

$$+\int\limits_{|\alpha_1|\geq r/\sqrt{2}}\frac{3}{\alpha_1^2}d\alpha_1.\int\limits_{-\infty}^{\infty}\left|\int\limits_{0}^{1} w(0,x_2)\sin(\alpha_2 x_2)dx_2\right|^2 d\alpha_2$$

$$= \frac{12\pi^2}{r^2}\left\|\frac{\partial w}{\partial x_1}\right\|_{L^2(\Omega)}^2 + \frac{6\sqrt{2}\pi}{r}\left\|w(1,.)\right\|_{L^2(0,1)}^2 + \frac{6\sqrt{2}\pi}{r}\left\|w(0,.)\right\|_{L^2(0,1)}^2$$

Noting that

$$w(1,x_2) = \int\limits_{0}^{1}\frac{\partial}{\partial x_1}\left(x_1 w(x)\right)dx_1 = \int\limits_{0}^{1}\left(w(x) + x_1\frac{\partial w}{\partial x_1}(x)\right)dx_1$$

, we get

$$|w(1,x_2)|^2 \leq \int\limits_{0}^{1}\left(2|w(x)|^2 + 2\left|\frac{\partial w}{\partial x_1}(x)\right|^2\right)dx_1$$

Hence,

$$\int\limits_{0}^{1}|w(1,x_2)|^2 dx_2 \leq 2\|w\|_{L^2(\Omega)}^2 + 2\left\|\frac{\partial w}{\partial x_1}\right\|_{L^2(\Omega)}^2$$

Similarly,

$$\int_0^1 |w(0,x_2)|^2\,dx_2 = \int_0^1 \left| \int_0^1 \frac{\partial}{\partial x_1}\left((1-x_1)w(x)\right)dx_1 \right|^2 dx_2$$
$$\leq \int_0^1\int_0^1 \left(2\,|w(x)|^2 + 2\left|\frac{\partial w}{\partial x_1}(x)\right|^2\right)dx_1dx_2 = 2\,\|w\|^2_{L^2(\Omega)} + 2\left\|\frac{\partial w}{\partial x_1}\right\|^2_{L^2(\Omega)}$$

Thus, we have

$$\int_{|\alpha_1|\geq r/\sqrt{2}} \left| \int_\Omega w(x)\cos(\alpha\cdot x)dx \right|^2 d\alpha \leq \frac{12\pi^2}{r^2}\left\|\frac{\partial w}{\partial x_1}\right\|_{L^2(\Omega)} +$$
$$+\frac{24\sqrt{2}\pi}{r}\left(\|w(1,.)\|^2_{L^2(\Omega)} + \left\|\frac{\partial w}{\partial x_1}\right\|^2_{L^2(\Omega)}\right) \leq \frac{24\sqrt{2}\pi}{r}\left(\|w(1,.)\|^2_{L^2(\Omega)} + 2\left\|\frac{\partial w}{\partial x_1}\right\|^2_{L^2(\Omega)}\right)$$

The proof is completed. □

Remark 1. *By the same way, we can show that, if $w \in H^1(\Omega)$ and $r > \pi/(2\sqrt{2})$ then*

$$\int_{R^2\backslash B(0,r)} \left| \int_Q w(x_1,x_2)\cos(\alpha_1 x_1)\cos(\alpha_2 x_2)dx \right|^2 d\alpha \leq \frac{16\sqrt{2}\pi}{r}\,\|w\|^2_{H^1(Q)}$$

This result improves immediately the results of [4].

Proof of theorem 2

Proof. Recall that $q, \delta_\varepsilon, R_\varepsilon$ are defined by (7), and $G_j(I_\varepsilon)$, B_ε are defined by Lemma 4. For all $j \in \{1,2\}$, we define $f_{j\varepsilon} : R^2 \to R$

$$f_{j\varepsilon}(\xi) = \frac{1}{4\pi^2}\int_{R^2} G_j(I_\varepsilon)(\alpha)e^{i(\xi\cdot\alpha)}d\alpha$$

Applying Lemma 4, we have $G_j(I_\varepsilon) \in L^1(R^2)\cap L^2(R^2)$, so $f_{j\varepsilon} \in C(R^2)\cap L^2(R^2)$ and $F(f_{j\varepsilon}) = G_j(I_\varepsilon)$. Applying Lemma 4 again, for all $\varepsilon \in (0, e^{-e})$, we get

$$\begin{aligned}\left|F(f_{j\varepsilon}) - F(\widetilde{f}_{jex})\right| &\leq \chi(B(0,R_\varepsilon))C_0R_\varepsilon\varepsilon^{\frac{1-6q}{2}} \\ &+2\chi(B_\varepsilon)\,\|f_{jex}\|_{L^2(\Omega)} + \chi(R^2\backslash B(0,R_\varepsilon))\left|F(\widetilde{f}_{jex})\right|\end{aligned} \tag{17}$$

where C_0 is a positive constant depending only I_{ex}. It implies that

$$\left|F(f_{j\varepsilon}) - F(\widetilde{f}_{jex})\right|^2 \le 2\chi(B(0,R_\varepsilon))C_0^2 R_\varepsilon^2 \varepsilon^{1-6q}$$
$$+4\chi(B_\varepsilon)\left\|f_{jex}\right\|^2_{L^2(\Omega)} + 2\chi(R^2\backslash B(0,R_\varepsilon))\left|F(\widetilde{f}_{jex})\right|^2$$

Hence,

$$\left\|F(f_{j\varepsilon}) - F(\widetilde{f}_{jex})\right\|^2_{L^2(R^2)} \le 2C_0^2\pi R_\varepsilon^4 \varepsilon^{1-6q} + 4m(B_\varepsilon)\left\|f_{jex}\right\|^2_{L^2(\Omega)} + 2\int\limits_{R^2\backslash B(0,R_\varepsilon)} \left|F(\widetilde{f}_{jex})\right|^2 d\alpha$$

It is obvious that $2C_0^2\pi R_\varepsilon^4\varepsilon^{1-6q} \le R_\varepsilon^{-1}$ for $\varepsilon > 0$ small enough. Moreover, since

$$B_\varepsilon \subset \left(\{\alpha \in B(0,R_\varepsilon), |D_1(I_{ex})(\alpha)| \le \varepsilon^q\} \cup \{\alpha \in B(0,R_\varepsilon), |D_2(I_{ex})(\alpha)| \le \varepsilon^q\}\right)$$

, we apply Lemma 2 (with $\varphi_0(t) = \varphi_{ex}(T-t)$) to get that $m(B_\varepsilon) \le 2R_\varepsilon^{-1}$ for $\varepsilon > 0$ small enough. Thus, for $\varepsilon > 0$ small enough, we get

$$\left\|F(f_{j\varepsilon}) - F(\widetilde{f}_{jex})\right\|^2_{L^2(R^2)} \le \frac{1}{R_\varepsilon} + \frac{8}{R_\varepsilon}\left\|f_{jex}\right\|^2_{L^2(\Omega)} + 2\int\limits_{R^2\backslash B(0,R_\varepsilon)} \left|F(\widetilde{f}_{jex})\right|^2 d\alpha d\beta$$

By Parseval equality, we have

$$\begin{aligned}\left\|f_{j\varepsilon} - f_{jex}\right\|^2_{L^2(\Omega)} &\le \left\|f_{j\varepsilon} - \widetilde{f}_{jex}\right\|^2_{L^2(R^2)} = \frac{1}{4\pi^2}\left\|F(f_{j\varepsilon}) - F(\widetilde{f}_{jex})\right\|^2_{L^2(R^2)} \\ &\le \frac{1}{4\pi^2}\left(\frac{1}{R_\varepsilon} + \frac{8}{R_\varepsilon}\left\|f_{jex}\right\|^2_{L^2(\Omega)} + 2\int\limits_{R^2\backslash B(0,R_\varepsilon)} \left|F(\widetilde{f}_{jex})\right|^2 d\alpha\right)\end{aligned} \tag{18}$$

for $\varepsilon > 0$ small enough. Since $F(\widetilde{f}_{jex}) \in L^2(R^2)$, we obtain that

$$\lim_{\varepsilon\to 0}\left\|f_{j\varepsilon} - f_{jex}\right\|_{L^2(\Omega)} = 0$$

If $f_{jex} \in H^1(\Omega)$ then using (18) and Lemma 6, we get

$$\begin{aligned}\left\|f_{j\varepsilon} - f_{jex}\right\|^2_{L^2(\Omega)} &\le \frac{1}{4\pi^2}\left(\frac{1}{R_\varepsilon} + \frac{8}{R_\varepsilon}\left\|f_{jex}\right\|^2_{L^2(\Omega)} + 2.4.\frac{72\sqrt{2}\pi}{R_\varepsilon}\left\|f_{jex}\right\|^2_{H^1(\Omega)}\right) \\ &\le \left(66\left\|f_{jex}\right\|^2_{H^1(\Omega)} + \frac{1}{4\pi^2}\right).\frac{1}{R_\varepsilon} = 63eT\left(66\left\|f_{jex}\right\|^2_{H^1(\Omega)} + \frac{1}{4\pi^2}\right).\frac{\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\end{aligned}$$

for $\varepsilon > 0$ small enough. This complete the proof. $\square$

Proof of Theorem 3

Proof. We shall use the notations of the proof of Theorem 2. Notice that the assumtion

$$\int\limits_{R^2} \left| \int\limits_{\Omega} f_{jex}(x). \cos(\alpha \cdot x) dx \right| d\alpha < \infty,$$

is equivalent to $F(\widetilde{f}_{jex}) \in L^1(R^2)$. Since $\widetilde{f}_{jex}, F(\widetilde{f}_{jex}) \in L^1(R^2) \cap L^2(R^2)$, we get

$$\widetilde{f}_{jex}(\xi) = \frac{1}{4\pi^2} \int\limits_{R^2} F(\widetilde{f}_{jex})(\alpha) e^{i(\alpha \cdot \xi)} d\alpha$$

Therefore,

$$4\pi^2 \left\| f_{j\varepsilon} - f_{jex} \right\|_{L^\infty(\Omega)} \le 4\pi^2 \left\| f_{j\varepsilon} - \widetilde{f}_{jex} \right\|_{L^\infty(R^2)} \le \left\| F(f_{j\varepsilon}) - F(\widetilde{f}_{jex}) \right\|_{L^1(R^2)} \tag{19}$$

From (17), we have

$$\left\| F(f_{j\varepsilon}) - F(\widetilde{f}_{jex}) \right\|_{L^1(R^2)} \le C_0 \pi R_\varepsilon^3 \varepsilon^{\frac{1-3q}{2}} + 2m(B_\varepsilon) \left\| f_{jex} \right\|_{L^2(\Omega)} + \int\limits_{R^2 \setminus B(0,R_\varepsilon)} \left| F(\widetilde{f}_{jex}) \right| d\alpha$$

For $\varepsilon > 0$ small enough, we have $C_0 \pi R_\varepsilon^3 \varepsilon^{\frac{1-3q}{2}} \le R_\varepsilon^{-1}$ and $m(B_\varepsilon) \le 2R_\varepsilon^{-1}$. Thus, from (19), for $\varepsilon > 0$ small enough, for all $j \in \{1, 2\}$, we get

$$4\pi^2 \left\| f_{j\varepsilon} - f_{jex} \right\|_{L^\infty(\Omega)} \le \frac{1}{R_\varepsilon} + \frac{4}{R_\varepsilon} \left\| f_{jex}) \right\|_{L^2(\Omega)} + \int\limits_{R^2 \setminus B(0,R_\varepsilon)} \left| F(\widetilde{f}_{jex}) \right| d\alpha$$

Since $F(\widetilde{f}_{jex}) \in L^1(R^2)$, we obtain that $\lim\limits_{\varepsilon \to 0} \left\| f_{j\varepsilon} - f_{jex} \right\|_{L^\infty(\Omega)} = 0$ for all $j \in \{1, 2\}$. □

Remark 2. *We can replace R_ε defined by (7) by*

$$\widetilde{R}_\varepsilon = 10 \left(\ln(\varepsilon^{-1}) \right)^{9/10}$$

to construct a better regularized solution in the case that ε is not too small.

4. A numerical experience

Assume that $T = 1$, $\mu = 1/12$, $\lambda = -1/8$.
We consider the exact data $I_{ex} = (\varphi, X, u_0, u_0^*, u_T)$ given by

$$\begin{aligned}
&\varphi = \frac{\pi^2}{3} \sin(\pi t), \\
&X_1 = \frac{\pi}{6} \sin(\pi t). \left[\sin(2\pi x_2) n_1 + \sin(4\pi x_1) n_2 \right], \\
&X_2 = \frac{\pi}{6} \sin(\pi t). \left[\sin(2\pi x_1) n_2 + \sin(4\pi x_2) n_1 \right], \\
&u_0 = u_T = (0, 0), \\
&u_0^* = \left(\pi \sin(4\pi x_1) \sin(2\pi x_2), \pi \sin(2\pi x_1) \sin(4\pi x_2) \right).
\end{aligned}$$

Then the corresponding exact solution of the system $(1)-(4)$ is

$$
\begin{aligned}
u_{ex} &= (\sin(\pi t)\sin(4\pi x_1)\sin(2\pi x_2), \sin(\pi t)\sin(4\pi x_1)\sin(2\pi x_2))\,,\\
f_{ex} &= (\cos(2\pi x_1)\cos(4\pi x_2), \cos(4\pi x_1)\cos(2\pi x_2))\,.
\end{aligned}
$$

For each $n = 1, 2, 3, ...$, we consider the inexact data $I_n = (\varphi_n, X^n, u_0^n, u_0^{*n}, u_T^n)$ given by

$$
\begin{aligned}
&\varphi_n = \varphi,\\
&X_1^n = X_1 + \frac{\pi}{12\sqrt{n}}\sin(\pi t).\left[\sin(2n\pi x_2)n_1 + 2\sin(2n\pi x_1)n_2\right],\\
&X_2^n = X_2 + \frac{\pi}{12\sqrt{n}}\sin(\pi t).\left[\sin(2n\pi x_1)n_2 + 2\sin(2n\pi x_2)n_1\right],\\
&u_0^n = u_T^n = (0,0),\\
&u_0^{*n} = u_0^* + \frac{\pi}{n\sqrt{n}}\sin(2n\pi x_1)\sin(2n\pi x_2)\,(1,1)\,.
\end{aligned}
$$

Then the corresponding disturbed solution of the system $(1)-(4)$ is

$$
\begin{aligned}
&u^n = u_{ex} + \frac{1}{n\sqrt{n}}\sin(\pi t)\sin(2n\pi x_1)\sin(2n\pi x_2)\,(1,1)\,,\\
&f_{di}^n = f_{ex} + \left[\left(\frac{3}{2}\sqrt{n} - \frac{3}{n\sqrt{n}}\right)\sin(2n\pi x_1)\sin(2n\pi x_2) + \frac{\sqrt{n}}{2}\cos(2n\pi x_1)\cos(2n\pi x_2)\right](1,1)\,.
\end{aligned}
$$

We get

$$
\begin{aligned}
&\varphi_n = \varphi,\\
&\left\|X_j^n(t,.) - X_j^{ex}(t,.)\right\|_{L^1(0,T,\partial\Omega)} = \frac{2}{\pi\sqrt{n}},\\
&u_0^n = u_0, u_T^n = u_T,\\
&\left\|u_{0j}^{*n} - u_{0j}^*\right\|_{L^1(\Omega)} = \frac{4}{\pi n\sqrt{n}}, \forall j \in \{1,2\},
\end{aligned}
$$

and

$$
\left\|f_{jdi}^n - f_{jex}\right\|^2_{L^2(\Omega)} = \frac{5}{8}n - \frac{9}{4n} + \frac{9}{4n^3}.
$$

Hence, when n is large, a small error of data will cause a large error of solution. It show that the problem is ill$-$posed, and a regularization is necessary.

We shall construct the regularized solution as in Theorem 1 corresponding $\varepsilon = n^{-1/2}$. From the straightforward calculation, we obtain that

$$
\begin{aligned}
D(I_n)(\alpha) &= \frac{32\pi^6\sin\left(\frac{|\alpha|}{2\sqrt{6}}\right)\sin\left(\frac{|\alpha|}{2\sqrt{3}}\right)}{\left(|\alpha|^2 - 24\pi^2\right).\left(|\alpha|^2 - 12\pi^2\right)},\\
g_1(I_n)(\alpha) &= D(I_n)(\alpha) \times (\sin(\alpha_1)\sin(\alpha_2) - (1-\cos(\alpha_1))(1-\cos(\alpha_2)))\times\\
&\quad\times\left(\frac{2\alpha_1\alpha_2}{(\alpha_1^2 - 4\pi^2)(\alpha_2^2 - 16\pi^2)} + \frac{\sqrt{n}(\alpha_1\alpha_2 + 12\pi^2(2-n^2))}{(\alpha_1^2 - 4n^2\pi^2)(\alpha_2^2 - 4n^2\pi^2)}\right).
\end{aligned}
$$

Thus, the regularized solution defined by

$$f_{1re}^n(x) = \frac{1}{4\pi^2} \int\limits_{B(0,\widetilde{R}_n)} \frac{g_1(I_n)(\alpha).D(I_n)(\alpha)}{\delta_n + (D(I_n)(\alpha))^2}.\cos(\alpha \cdot x) d\alpha,$$

where

$$\delta_n = n^{-13/28}, \widetilde{R}_n = 10 \left(\ln(\sqrt{n})\right)^{9/10}.$$

For example, if $\varepsilon = 10^{-2}$ then

$$n = 10^4, \delta_n = 0.01389495494, \widetilde{R}_n = 39.52948133,$$

and we have some figures about the exact solution f_{1ex}, the disturbed solution f_{1di}^n and the regularized solutionf_{1re}^n.

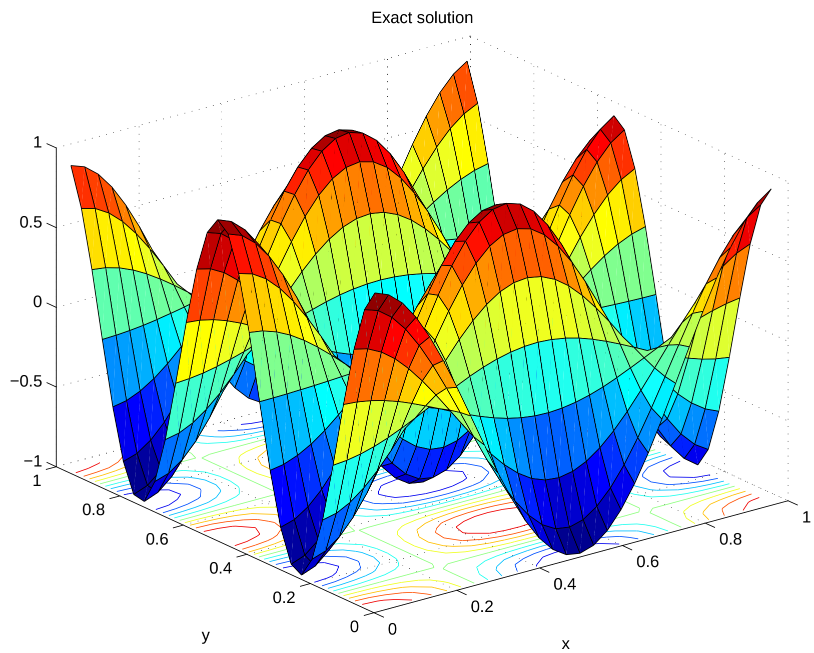

Figure 1. The exact solution.

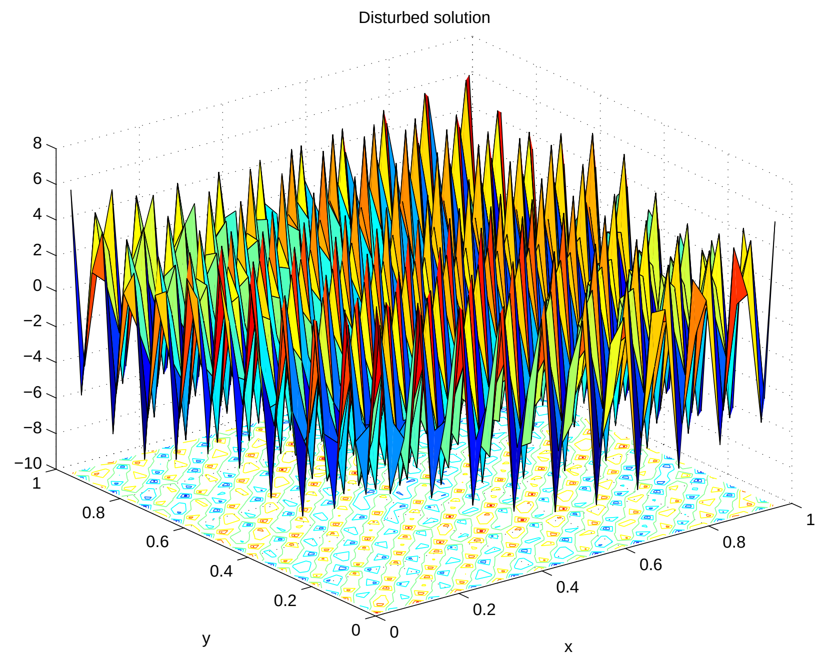

Figure 2. The disturbed solution.

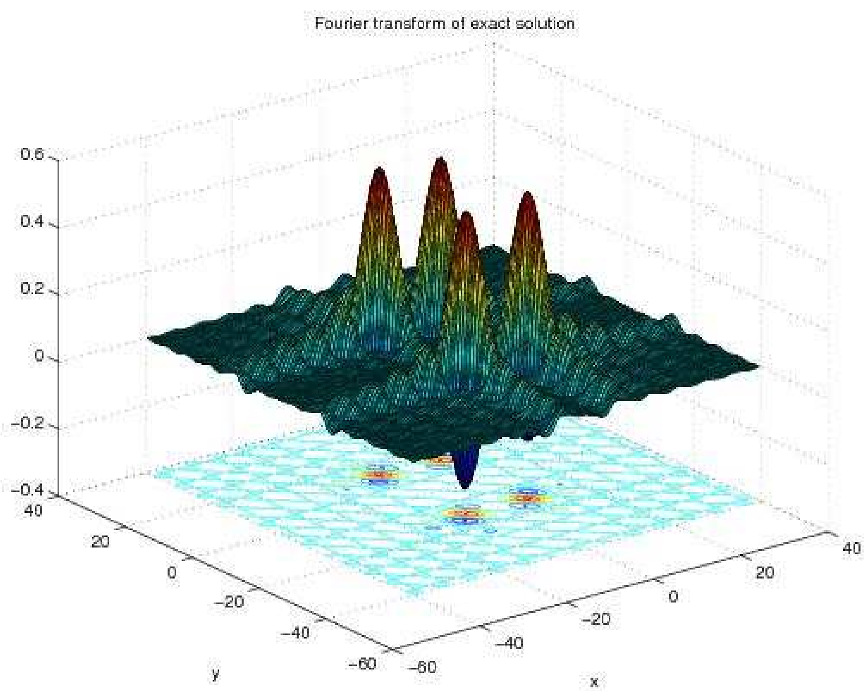

Figure 3. The Fourier transform of the exact solution.

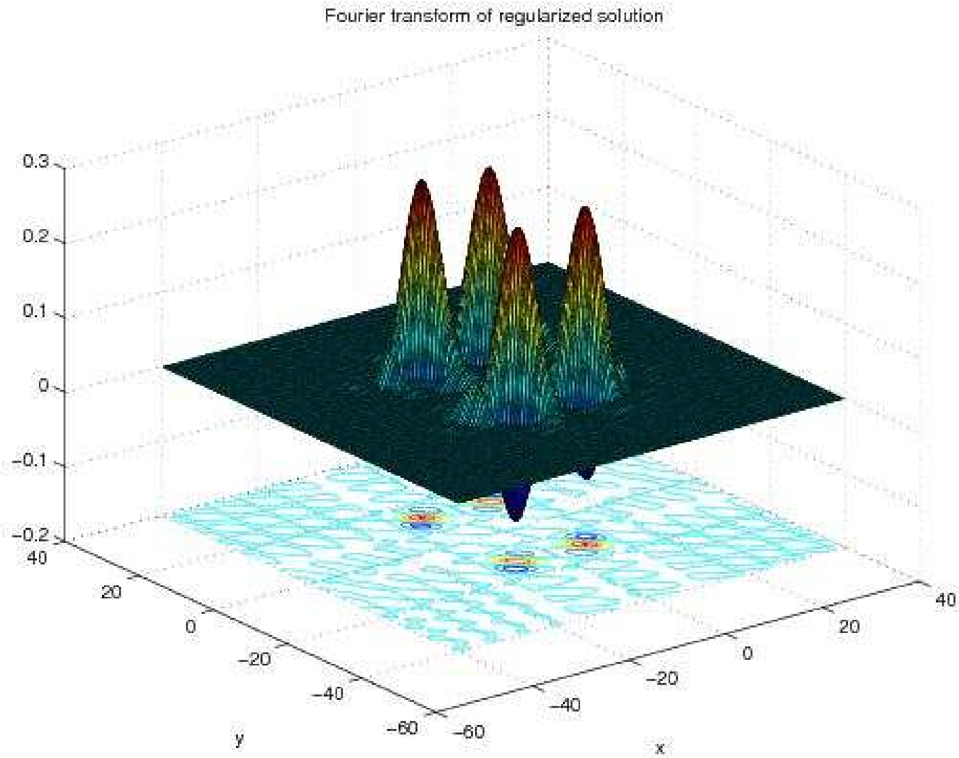

Figure 4. The Fourier transform of the regularized solution.

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